# Learning Monotone Nonlinear Models using the Choquet Integral

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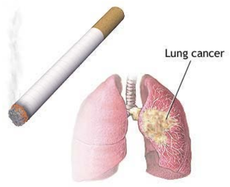
## Monotonicity

Incorporating background knowledge, such as ***monotonicity***, into the learning process is an important aspect in machine learning research.




For example, the higher the tobacco consumption, the more likely a patient suffers a lung cancer.

Monotonicity is easy to ensure for a linear model but harder to ensure for a non-linear one.

## Additive & Non-Additive Measures

Let $C = \{c_1, \ldots, c_m\}$ be a finite set and $\mu(\cdot)$ a measure $2^C \to [0,1]$. For each $A \subseteq C$, we interpret $\mu(A)$ as the *weight* of the set $A$.

$C$ = {speaking Chinese, coding in Java, coding in C}

For an additive measure:
$\mu(A \cup B) = \mu(A) + \mu(B),\ \forall A, B \subseteq C$ such that $A \cap B = \emptyset$.

$\mu$ ({speaking Chinese}) = 0.2 — $\mu$({speaking Chinese, coding in Java}) = 0.6
$\mu$ ({coding in Java}) = 0.4 — $\mu$({speaking Chinese, coding in C}) = 0.6
$\mu$ ({coding in C}) = 0.4 — $\mu(C)$ = 1

A (non-additive) measure is normalized and monotone:
$\mu(\emptyset) = 0,\ \mu(C) = 1$, and $\mu(A) \leq \mu(B) \quad \forall\, A \subseteq B \subseteq C$.

$\mu$ ({speaking Chinese}) = 0 — $\mu$({speaking Chinese, coding in Java}) = 1
$\mu$ ({coding in Java}) = 0 — $\mu$({speaking Chinese, coding in C}) = 0.7
$\mu$ ({coding in C}) = 0 — $\mu(C)$ = 1

## Importance of Criteria & Interaction

For an additive measure:

- There is no possibility to model interaction between criteria.
- $\mu(\{c_i\})$ is a natural quantification of the importance of $c_i$.

For a non-additive measure:

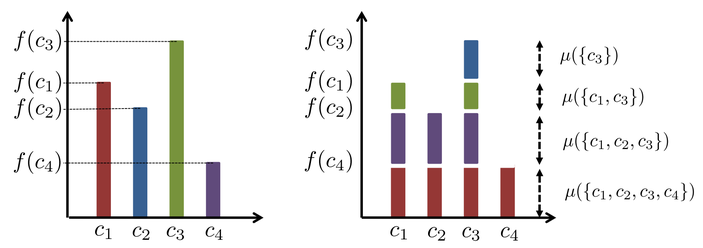
- Importance of criteria can be measured by the Shapley index:

$$\varphi(c_i) = \sum_{A \subseteq C \setminus \{c_i\}} \frac{1}{m\binom{m-1}{|A|}} \Big(\mu(A \cup \{c_i\}) - \mu(A)\Big) .$$

- Interactions between criteria can be measured by the interaction index:

$$I_{i,j} = \sum_{A \subseteq C \setminus \{c_i, c_j\}} \frac{\mu(A \cup \{c_i, c_j\}) - \mu(A \cup \{c_i\}) - \mu(A \cup \{c_j\}) + \mu(A)}{(m-1)\binom{m-2}{|A|}} .$$

## Discrete Choquet Integral: A Brief Intro




The **discrete Choquet integral** of $f : C \to \mathbb{R}_+$ with respect to $\mu$ is defined as follows:

$$\mathcal{C}_\mu(f) = \sum_{i=1}^{m} \big(f(c_{(i)}) - f(c_{(i-1)})\big) \cdot \mu(A_{(i)}) ,$$

where $(\cdot)$ is a permutation of $\{1, \ldots, m\}$ such that $0 \leq f(c_{(1)}) \leq f(c_{(2)}) \leq \ldots \leq f(c_{(m)})$, and $A_{(i)} = \{c_{(i)}, \ldots, c_{(m)}\}$.

In our case, $f(c_i) = x_i$ is the value of the $i$-th variable.

## From Logistic to Choquistic Regression

**Logistic** $\mathbf{P}(y = 1 \,|\, \boldsymbol{x}) = \Big(1 + \exp\big(\ -w_0 - \boldsymbol{w}^\top \boldsymbol{x}\ \big)\Big)^{-1}$

**Choquistic** $\mathbf{P}(y = 1 \,|\, \boldsymbol{x}) = \Big(1 + \exp\big(\ -\gamma\,(\mathcal{C}_\mu(\boldsymbol{x}) - \beta)\ \big)\Big)^{-1}$

Choquet integral of (normalized) attribute values

- It can be shown that, by choosing the parameters in a proper way, logistic regression is indeed **a special case of choquistic regression**.

## Choquistic Regression: Interpretation

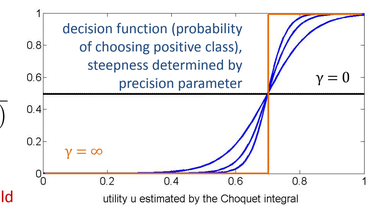
Interpretation of choquistic regression as a **two-stage process**:

(1) a (latent) utility degree $u = \mathcal{C}_\mu(\boldsymbol{x}) \in [0,1]$ is determined by the Choquet integral
(2) a discrete choice is made by thresholding $u$ at $\beta$

**Thresholding:**

$$\mathbf{P}(y = 1) = \frac{1}{1 + \exp\big(\ -\gamma\,(\mathcal{C}_\mu(\boldsymbol{x}) - \beta)\ \big)}$$

precision of the model — utility threshold




## Choquistic Regression: Parameter Estimation

ML estimation leads to a **constrained optimization problem**:

$$\min_{\boldsymbol{m},\gamma,\beta}\ \gamma \sum_{i=1}^{n} (1 - y^{(i)})\,(\mathcal{C}_{\boldsymbol{m}}(\boldsymbol{x}^{(i)}) - \beta) + \sum_{i=1}^{n} \log\Big(1 + \exp(-\gamma\,(\mathcal{C}_{\boldsymbol{m}}(\boldsymbol{x}^{(i)}) - \beta))\Big)$$

subject to:

$0 \leq \beta \leq 1$
$0 < \gamma$
} conditions on utility threshold and precision

$\sum_{T \subseteq C} \boldsymbol{m}(T) = 1$
$\sum_{B \subseteq A \setminus \{c_i\}} \boldsymbol{m}(B \cup \{c_i\}) \geq 0 \quad \forall A \subseteq C, \forall c_i \in C$
} normalization and monotonicity of the non-additive measure

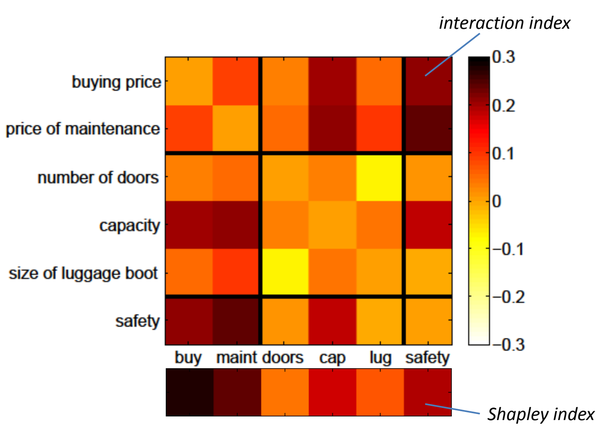
→ solution with sequential quadratic programming

## Experimental Evaluation

| | dataset | CR ■■ | LR ■ | KLR-ply ■ | KLR-rbf ■ | MORE ■■ |
|---|---|---|---|---|---|---|
| 20% | DBS | .2226±.0380 (4) | .1803±.0336 (1) | .2067±.0447 (3) | .1922±.0501 (2) | .2541±.0142 (5) |
| | CPU | .0457±.0338 (2) | .0430±.0318 (1) | .0586±.0203 (3) | .0674±.0276 (4) | .1033±.0681 (5) |
| | BCC | .2939±.0100 (4) | .2761±.0265 (1) | .3102±.0386 (5) | .2859±.0329 (3) | .2781±.0219 (2) |
| | MPG | .0688±.0098 (2) | .0664±.0162 (1) | .0729±.0116 (4) | .0705±.0122 (3) | .0800±.0198 (5) |
| | ESL | .0764±.0291 (3) | .0747±.0243 (1) | .0752±.0117 (2) | .0794±.0134 (4) | .1035±.0332 (5) |
| | MMG | .1816±.0140 (3) | .1752±.0106 (2) | .1970±.0095 (4) | .2011±.0123 (5) | .1670±.0120 (1) |
| | ERA | .2997±.0123 (2) | .2922±.0096 (1) | .3011±.0132 (3) | .3259±.0172 (5) | .3040±.0192 (4) |
| | LEV | .1527±.0138 (1) | .1644±.0106 (4) | .1570±.0116 (2) | .1577±.0124 (3) | .1878±.0242 (5) |
| | CEV | .0441±.0128 (1) | .1689±.0066 (5) | .0571±.0078 (3) | .0522±.0085 (2) | .0690±.0408 (4) |
| | avg. rank | **2.4** | **1.9** | **3.3** | **3.4** | **4** |
| 50% | DBS | .1560±.0405 (3) | .1443±.0371 (2) | .1845±.0347 (5) | .1628±.0269 (4) | .1358±.0432 (1) |
| | CPU | .0156±.0135 (1) | .0400±.0106 (3) | .0377±.0153 (2) | .0442±.0223 (5) | .0417±.0198 (4) |
| | BCC | .2871±.0358 (4) | .2647±.0267 (2) | .2706±.0295 (3) | .2879±.0269 (5) | .2616±.0320 (1) |
| | MPG | .0641±.0175 (1) | .0684±.0206 (2) | .1462±.0218 (5) | .1361±.0197 (4) | .0700±.0162 (3) |
| | ESL | .0660±.0135 (1) | .0697±.0144 (3) | .0704±.0128 (5) | .0699±.0148 (4) | .0690±.0171 (2) |
| | MMG | .1736±.0157 (3) | .1710±.0161 (2) | .1859±.0141 (4) | .1900±.0169 (5) | .1604±.0139 (1) |
| | ERA | .3008±.0135 (3) | .3054±.0140 (4) | .2907±.0136 (1) | .3084±.0152 (5) | .2928±.0168 (2) |
| | LEV | .1357±.0122 (1) | .1641±.0131 (4) | .1500±.0098 (3) | .1482±.0112 (2) | .1658±.0202 (5) |
| | CEV | .0346±.0076 (1) | .1667±.0093 (5) | .0357±.0113 (2) | .0393±.0090 (3) | .0443±.0080 (4) |
| | avg. rank | **2** | **3** | **3.3** | **4.1** | **2.6** |
| 80% | DBS | .1363±.0380 (2) | .1409±.0336 (4) | .1422±.0498 (5) | .1386±.0521 (3) | .0974±.0560 (1) |
| | CPU | .0089±.0126 (1) | .0366±.0068 (4) | .0329±.0295 (2) | .0384±.0326 (5) | .0342±.0232 (3) |
| | BCC | .2631±.0424 (2) | .2669±.0483 (3) | .2784±.0277 (4) | .2937±.0297 (5) | .2526±.0472 (1) |
| | MPG | .0526±.0263 (1) | .0538±.0282 (2) | .0669±.0251 (4) | .0814±.0309 (5) | .0656±.0248 (3) |
| | ESL | .0517±.0235 (1) | .0602±.0264 (2) | .0654±.0228 (3) | .0718±.0188 (5) | .0657±.0251 (4) |
| | MMG | .1584±.0255 (2) | .1683±.0231 (3) | .1798±.0293 (4) | .1853±.0232 (5) | .1521±.0249 (1) |
| | ERA | .2855±.0257 (1) | .2932±.0261 (4) | .2885±.0302 (2) | .2951±.0286 (5) | .2894±.0278 (3) |
| | LEV | .1312±.0186 (1) | .1662±.0171 (5) | .1518±.0104 (3) | .1390±.0129 (2) | .1562±.0252 (4) |
| | CEV | .0221±.0091 (1) | .1643±.0184 (5) | .0376±.0091 (3) | .0262±.0067 (2) | .0408±.0090 (4) |
| | avg. rank | **1.3** | **3.6** | **3.3** | **4.1** | **2.7** |

■ monotone classifier ■ nonlinear classifier

## Importance & Interactions (Car Evaluation)




## Conclusions & Outlook

We advocate the use of the discrete Choquet integral as an aggregation operator in machine learning, especially in learning monotone models.

As a concrete application, we have proposed choquistic regression, a generalization of logistic regression.

First experimental results confirm advantages of the Choquet integral.

**Ongoing work:** Restriction to k-additive measures, for a properly chosen k
–full flexibility is normally not needed and may even lead to overfitting the data
–advantages from a computational point of view
–key question: how to find a suitable k in an efficient way?

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